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The Ovals of the Plane Sextic Curve.

BY J. EDMUND WRIGHT.

It is known that the plane sextic without double points has at most eleven separate ovals, and Hilbert has stated, though without proof, that these cannot all lie external to one another. No proof of this theorem has in fact yet been given, though Miss Ragsdale* has shown that a sextic with the maximum number of ovals, all external, cannot be obtained by the ordinary processes whereby curves with the maximum number of circuits are derived. This paper gives a proof of the theorem.

Let it be assumed that $u = 0$ is a sextic with eleven external ovals all lying in the finite part of the plane, and suppose that u is chosen to be positive at infinity, then u is negative inside each oval, and positive outside. Now consider $u + c = 0$ where c is a positive constant. As c increases from zero, each oval gradually shrinks up. This process of shrinking continues either until one of the ovals reduces to an isolated point, or until an oval meets itself again, thus giving an ordinary node on that oval. This last case is however impossible, for if it were possible a slight further increase in c would give a sextic with twelve ovals. A sextic, even though reducible, cannot have twelve ovals. Let then $u + c_1 \equiv v = 0$ be the sextic with one isolated point and ten external ovals, no one of which contains this point. Now let A and B be two real straight lines through the point, and consider the sextic $v + k(A^2 + B^2) = 0$ where k is a positive constant. The particular point obtained, as k increases from zero, remains an isolated point on the sextic, and again the ovals shrink up until as before one reduces to an isolated point. Let k_1 be the value of k for this isolated point, and consider the sextic $w \equiv v + k_1 (A^2 + B^2) = 0$. This sextic has nine external ovals, and two isolated points both lying outside all the ovals. Through the two isolated points may be described three real conics with no other common points, say L, M, N. Then, as before, we consider the sextic

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* American Journal of Mathematics (1906). *On the Arrangement of the real Branches of Plane Algebraic Curves.*

$$w + k\,(L^2 + M^2 + N^2) = 0,$$

and thus obtain a sextic with three isolated points. Now suppose that we have obtained a sextic with r isolated points, and $11 - r$ external ovals, and suppose r is less than eight, we can describe through the r points three real cubics C_1, C_2, C_3, which have no other common points, and therefore if $U = 0$ be the sextic, $U + k\,(C_1^2 + C_2^2 + C_3^2) = 0$ leads to a sextic with $r + 1$ isolated points and $10 - r$ ovals, all external to each other. If $r = 8$, two real cubics may be described through the eight points, and they have one other point common. This other point can lie inside one at most of the remaining ovals, and therefore $U + k\,(C_1^2 + C_2^2) = 0$ will give a sextic with two ovals and nine external isolated points. It is clear that only one cubic can be described through these points, for if there were more there would be an infinity of them, and one of these could be made to pass through an arbitrary point on the sextic. The cubic and the sextic would then intersect in 19 points, which is impossible. Let then $U = 0$ be the sextic with nine points and two ovals, and let $C = 0$ be the cubic through the nine points. Then $U + k\,C^2 = 0$ is a sextic with the nine isolated points as double points, and C has at most one even circuit. Hence, since C can never meet the sextic again, only one at most of the two ovals of U can contain any part of C inside it,[*] and therefore, as before, by increasing k from zero a sextic can be obtained with ten isolated points and one oval external to the ten points. The sextic is now unicursal, and the theorem to be proved is that there can exist no unicursal sextic with ten isolated points and an external oval. The sextic thus reduces to a finite oval and ten external isolated points, each at a finite distance from the others and from the oval. Let $S = 0$ be this sextic.

Through eight of the points describe two cubics C_1, C_2 each passing through one of the other two points. Also let C_1 and C_2 be so chosen that the oval lies in the part of the plane where they are positive, and consider the sextic $S + k\,C_1 C_2 = 0$. As k increases from zero the eight points remain fixed isolated points and the two remaining points become ovals lying in the part of the plane for which $C_1 C_2$ is negative. Also the large oval shrinks up. Now suppose for any point on this oval C_1 and C_2 have the values d_1 and d_2 and $\sqrt{S_x^2 + S_y^2}$ the value p. Then if k increases from zero to a small positive value, the oval shrinks up a small amount, and the point considered moves inward a distance $k\frac{d_1 d_2}{p}$. Also suppose at the 9th point (on C_1), C_2 is δ_2, C_1 is $ax+by+\ldots.$ and S is $\alpha x^2+2\beta xy+\gamma y^2+\ldots$, where $x=0$ $y=0$ is the point considered. The small oval is practically the ellipse $\alpha x^2+2\beta xy+\gamma y^2+k\delta_2(ax+by)=0$, so that this ellipse has dimensions proportional to k, and it moves towards the oval only when a branch of C_1 or C_2 separates it from the oval. Hence if the oval is at a finite distance from C_1 and C_2 we can thus obtain a sextic with eight isolated points and three external ovals, and the first oval is made smaller than before. We now proceed as before to shrink up the three ovals and so obtain again ten points and one oval. If the original oval should be one of the two that reduce to points, we can arrange the second process so as to stop at a smaller value of k, and thus by choosing k properly have the three ovals all points. This would give a sextic with eleven isolated points. The process must be stopped if the sextic approaches to within a small distance of one of the cubics. If however a part of the oval is at a distance $<\varepsilon$ from say C_1 it must be possible to write $S=C_1C'+\varepsilon S'$, where C' is a cubic and S' is a sextic with finite coefficients. But obviously a sextic such as that considered cannot be obtained from any two cubics and the only possibility is that C_1 consists of a straight line and a conic. If C_1 consists of a straight line and a conic, then there remains to be considered the possibility $S=PQ+\varepsilon S'$ where P is a conic and Q is a quartic, and the only possibility is that Q is a perfect square passing through the ten isolated points. This however is impossible since a conic cannot pass through more than six of the *d. p. s* of a sextic. Hence the oval is always at a finite distance away from the cubics and thus the process can be continued until the ovals reduce to eleven isolated *d. p. s.* There remains yet one possibility. The two *d. p. s* which are varied might run together. Suppose that the process is continued until the two variable points are at a very small distance apart, and then take one of these as one of the eight fixed points, allowing one of the first eight to be movable, and continue the process until these are a small distance apart if before this time the oval of the sextic cannot be reduced to a point. We can in this way keep seven points fixed and continue our process until the three points come close together. Now these points must in the limit form a point equivalent to three ordinary *d. p. s*, and it cannot be an ordinary triple point, for then a real branch of the sextic would necessarily pass through it.

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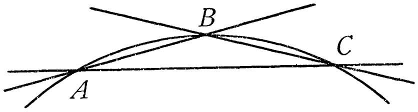
[*] If an oval of the cubic lies inside an oval of the sextic, that oval cannot shrink up.

It must therefore be of the type known as an oscnode, and a cubic passing through the three when they are at a small distance apart will have an ordinary point at the place. Describe the cubic through the three and six of the seven fixed points. Let the three special ones be A, B, C and consider the three cubics through BC, CA, AB, and the other seven points. Suppose that C_1 is the cubic through the three ABC, and C_2 is, say, that through BC. Then A and the point not on C_1 turn into small ovals which may possibly approach each other. But it is easily seen that the cubics BC and CA are close to each other, and therefore if we take C_2 as the cubic CA the ovals derived from B and the point not on C_1 will be separated from one another by C_2. They therefore cannot approach very close to each other without first separating. They cannot approach across C_2. If the branch between them be the even branch of C_2 they can never approach. If it be the odd branch they can only opproach across ∞y. Hence through the whole process we may always hold six points fixed and can continue indefinitely the alternate process outlined, always arranging that the four movable points shall not run together, and we finally obtain a sextic consisting of eleven isolated points. Such a sextic would, however, be reducible, and it may easily be verified that no such reducible sextic exists. Hence the theorem is proved that *a sextic cannot consist of eleven external ovals.*



I shall show in a later paper that the only possible arrangements for the ovals of a sextic are one internal and ten external, or ten internal and one external.

BRYN MAWR, *March* 1907.